

Lyapunov-type inequality for a fractional boundary value problem with natural conditions*

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Abstract

We derive a new Lyapunov type inequality for a boundary value problem involving both left Riemann–Liouville and right Caputo fractional derivatives in presence of natural conditions. Application to the corresponding eigenvalue problem is also discussed.

Keywords: fractional calculus, Lyapunov inequality, eigenvalue problem.

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1 Introduction

Lyapunov's inequality is a useful tool in the study of spectral properties of ordinary differential equations [5, 15]. The classical Lyapunov inequality is given in the following theorem.

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Theorem 1 (See [9, 12]). *If the boundary value problem*

$$\begin{aligned} -u''(t) &= q(t)u(t), \quad a < t < b, \\ u(a) &= u(b) = 0 \end{aligned} \quad (1)$$

has a nontrivial continuous solution, where q is a real and continuous function, then

$$\int_a^b |q(t)| dt \geq \frac{4}{b-a}. \quad (2)$$

Furthermore, the constant 4 in (2) is sharp.

Many authors have extended the Lyapunov inequality (2) [5, 11, 15, 16]. Here we are interested in generalizations of (2) that are associated to a fractional differential equation, where the second order derivative in (1) is substituted by some fractional operator [1, 6, 7, 8, 9, 10].

Recently, in 2016, Ferreira obtained Lyapunov type inequalities for Caputo or Riemann–Liouville sequential fractional differential equations with Dirichlet boundary conditions [8]. In 2017, Agarwal and Özbekler obtained Lyapunov type inequalities for mixed nonlinear Riemann–Liouville fractional differential equations with a forcing term and Dirichlet boundary conditions [1]. Here, we prove a new Lyapunov type inequality for a sequential fractional boundary value problem involving both Riemann–Liouville and Caputo fractional derivatives:

$$-{}^C D_{b-}^{\alpha} D_{a+}^{\beta} u(t) + q(t)u(t) = 0, \quad a < t < b, \quad (3)$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, ${}^C D_{b-}^{\alpha}$ denotes the right Caputo derivative, D_{a+}^{β} denotes the left Riemann–Liouville derivative, u is the unknown function, and q is continuous on $[a, b]$. Such problems, with both left and right fractional derivatives, arise in the study of Euler–Lagrange equations for fractional problems of the calculus of variations [3, 13, 14]. We consider the fractional differential equation (3) with natural boundary conditions [2, 4]:

$$u(a) = D_{a+}^{\beta} u(b) = 0. \quad (4)$$

To the best of our knowledge, this is the first work to give a Lyapunov type inequality (see Theorem 4) for mixed right Caputo and left Riemann–Liouville fractional differential equations. The result is important because, in many applications, the natural boundary conditions have a physical interpretation. For instance, fractional variational problems require imposition of natural boundary conditions that the optimum solution must satisfy [3]. Moreover, for boundary value problems, when sufficient kinematic conditions are not specified, the natural boundary conditions are necessary to solve the problem analytically [14]. Hence, natural boundary conditions are necessary to solve a fractional boundary value problem, and the fractional problem with natural boundary conditions is not obvious neither trivial. This is the case of the problem studied in our paper, where condition (4) is imposed naturally, because we have both a right Caputo derivative and a left Riemann–Liouville derivative in equation (3).

The paper is organized as follows. In Section 2, we briefly recall the necessary concepts and results from fractional calculus. Our results are then formulated and proved in Section 3. We end with Section 4, where an example of application to a fractional eigenvalue problem is given, and Section 5 of conclusion.

2 Preliminaries

We recall here the essential definitions on fractional calculus. For details on the subject we refer the reader to [11, 16]. Let $p > 0$. Then the left and right Riemann–Liouville fractional integral of a function g are defined respectively by

$$\begin{aligned} I_{a+}^p g(t) &= \frac{1}{\Gamma(p)} \int_a^t \frac{g(s)}{(t-s)^{1-p}} ds, \\ I_{b-}^p g(t) &= \frac{1}{\Gamma(p)} \int_t^b \frac{g(s)}{(s-t)^{1-p}} ds. \end{aligned}$$

The left Riemann–Liouville fractional derivative and the right Caputo fractional derivative of order $p > 0$ of a function g are

$$\begin{aligned} D_{a+}^p g(t) &= \frac{d^n}{dt^n} (I_{a+}^{n-p} g)(t), \\ {}^C D_{b-}^p g(t) &= (-1)^n I_{b-}^{n-p} g^{(n)}(t), \end{aligned}$$

respectively, where $p \in (n-1, n)$. With respect to the properties of Riemann–Liouville and Caputo fractional operators, we mention the following. Let $p \in (n-1, n)$ and $f \in L_1[a, b]$. Then,

1. $I_{a+}^p D_{a+}^p f(t) = f(t) - \sum_{i=1}^n c_i (t-a)^{p-i};$
2. $I_{b-}^p {}^C D_{b-}^p f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!} (b-t)^k.$

3 Lyapunov-type inequality

We begin by transforming problem (3)–(4) into an equivalent integral equation.

Lemma 2. *Assume that $0 < \alpha, \beta \leq 1$ and $1 < \alpha + \beta \leq 2$. Function u is a solution to the boundary value problem (3)–(4) if and only if u satisfies the integral equation*

$$u(t) = \int_a^b G(t, r) q(r) u(r) dr,$$

where

$$G(t, r) = \begin{cases} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^r (t-s)^{\beta-1} (r-s)^{\alpha-1} ds, & a \leq r \leq t \leq b, \\ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} (r-s)^{\alpha-1} ds, & a \leq t \leq r \leq b. \end{cases} \quad (5)$$

Proof. Applying the properties of Caputo and Riemann–Liouville fractional derivatives and the boundary conditions (4), then using the Fubini theorem, we obtain

$$\begin{aligned}
u(t) &= I_{a+}^{\beta} I_{b-}^{\alpha} q(t) u(t) \\
&= \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^t (t-s)^{\beta-1} \left(\int_s^b (r-s)^{\alpha-1} q(r) u(r) dr \right) ds \\
&= \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^t \left(\int_a^r (t-s)^{\beta-1} (r-s)^{\alpha-1} ds \right) q(r) u(r) dr \\
&\quad + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_t^b \left(\int_a^t (t-s)^{\beta-1} (r-s)^{\alpha-1} ds \right) q(r) u(r) dr,
\end{aligned}$$

from which the intended result follows. \square

We now prove some properties of the Green function (5).

Lemma 3. *Assume that $0 < \alpha, \beta \leq 1$ and $1 < \alpha + \beta \leq 2$. Then the Green function G defined by (5) satisfies the following properties:*

1. $G(t, r) \geq 0$ for all $a \leq r \leq t \leq b$;
2. $\max_{t \in [a, b]} G(t, r) = G(r, r)$ for all $r \in [a, b]$;
3. $\max_{r \in [a, b]} G(r, r) = \frac{(b-a)^{\alpha+\beta-1}}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}$.

Proof. Obviously, $G(t, r) \geq 0$ for $t, r \in (a, b)$. Set

$$\begin{aligned}
g_1(t, r) &= \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^r (t-s)^{\beta-1} (r-s)^{\alpha-1} ds, \quad a \leq r \leq t \leq b, \\
g_2(t, r) &= \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \left(\int_a^t (t-s)^{\beta-1} (r-s)^{\alpha-1} ds \right), \quad a \leq t \leq r \leq b.
\end{aligned}$$

For $r \leq t$, we have

$$g_1(t, r) \leq \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^r (r-s)^{\beta-1} (r-s)^{\alpha-1} ds = G(r, r). \quad (6)$$

Similarly, if $t \leq r$, then

$$\begin{aligned}
g_2(t, r) &\leq \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^t (t-s)^{\beta-1} (t-s)^{\alpha-1} ds \\
&= \frac{(t-a)^{\alpha+\beta-1}}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \\
&\leq \frac{(r-a)^{\alpha+\beta-1}}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \\
&= G(r, r).
\end{aligned} \quad (7)$$

Thus, from (6) and (7), we get $\max_{t \in [a, b]} G(t, r) = G(r, r)$ for all $r \in [a, b]$. Since $G(r, r)$ is increasing, we obtain that

$$\max_{r \in [a, b]} G(r, r) = \frac{(b-a)^{\alpha+\beta-1}}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}.$$

The proof is complete. \square

Now we are ready to give the Lyapunov type inequality for problem (3)–(4).

Theorem 4. *Assume that $0 < \alpha, \beta \leq 1$ and $1 < \alpha + \beta \leq 2$. If the fractional boundary value problem (3)–(4) has a nontrivial continuous solution, then*

$$\int_a^b |q(r)| dr \geq \frac{(\alpha + \beta - 1)\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha+\beta-1}}. \quad (8)$$

Furthermore, the inequality (8) is sharp.

Proof. From Lemma 3, we have

$$\begin{aligned} |u(t)| &\leq \int_a^b G(t, r) |q(r)| |u(r)| dr \\ &\leq \int_a^b G(r, r) |q(r)| |u(r)| dr \\ &\leq \frac{(b-a)^{\alpha+\beta-1} \|u\|}{(\alpha + \beta - 1)\Gamma(\alpha)\Gamma(\beta)} \int_a^b |q(r)| dr, \end{aligned}$$

where $\|u\| = \max_{t \in [a, b]} |u(t)|$. Consequently,

$$\|u\| \leq \frac{(b-a)^{\alpha+\beta-1} \|u\|}{(\alpha + \beta - 1)\Gamma(\alpha)\Gamma(\beta)} \int_a^b |q(r)| dr.$$

Thus, inequality (8) follows. \square

Next we give a Lyapunov type inequality in the case $\alpha = \beta = 1$.

Corollary 5. *If the boundary value problem*

$$\begin{aligned} u''(t) + q(t)u(t) &= 0 \\ u(a) = 0 &= u'(b) \end{aligned}$$

has a nontrivial continuous solution, then the Lyapunov inequality

$$\int_a^b |q(r)| dr \geq \frac{1}{(b-a)}$$

holds.

4 Application to a fractional eigenvalue problem

We end with an application of the Lyapunov-type inequality (8) to a fractional eigenvalue problem generated by the fractional differential equation

$${}^C D_{b-}^{\alpha} D_{a+}^{\beta} u(t) = \lambda u(t), \quad a < t < b, \quad \lambda \in \mathbb{R}, \quad (9)$$

subject to the boundary conditions (4).

Corollary 6. *Assume that $0 < \alpha, \beta \leq 1$ and $1 < \alpha + \beta \leq 2$. If λ is an eigenvalue to the fractional boundary value problem defined by (9) and (4), then*

$$|\lambda| \geq \frac{(\alpha + \beta - 1) \Gamma(\alpha) \Gamma(\beta)}{(b - a)^{\alpha + \beta}}.$$

5 Conclusion

We derived a new Lyapunov-type inequality for a sequential boundary value problem subject to natural boundary conditions. The idea of studying a differential equation depending on the sequence of right and left fractional derivatives, is relevant in applications and seems to be new. Contrary to existing papers on Lyapunov inequalities and its generalizations, here the expression of the Green function G is not classical and is expressed by integrals, which is nontrivial.

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